Design Considerations for Bent-Beam Waveguides

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Abstract—**The description of the beam-waveguide in terms of ray optics and wave optics is combined to describe the performance of waveguides with predetermined bending radii. We require that the beam in a bend of the beam-waveguide departs from its axis no more than an amount equal to half its spot size. This requirement is sufficient to determine uniquely the spacing and focal length of the lenses.**

It turns out that it is advantageous to space the lenses close to four times their focal length, in order to enable the waveguide to direct a light beam around sharp bends. However, transverse random displacements of the lenses also cause the beam to depart from the axis of the guide. This departure becomes very large if the lens spacing approaches four times the focal length. A guide which is designed to negotiate sharp bends is more seriously affected by random displacements of the lenses. Further analysis of imperfections and experimental work will be required before a final design choice can be made.

Introduction

THE INTEREST in light as a communications carrier has been revived with the invention of the laser. However, a light communications system becomes feasible only if a low-loss transmission medium can be designed. One possible solution to this problem is Goubau's beam-waveguide [1]. It consists of a sequence of positive lenses which are arranged in a straight line and keep the light beam from spreading apart by refocusing it periodically. The power loss due to diffraction at the lens apertures can be kept extremely low.

It is also well known [2] that the beam-waveguide is capable of leading the light beam around bends, Fig. 1. However, the light beam will follow an oscillatory trajectory after leaving the bend unless one takes special measures to avoid these beam undulations [2]. Every deviation from perfect straightness will, in general, cause the light beam to oscillate around the waveguide axis. If the amplitudes of these beam oscillations become too large the beam moves close to the edge of lenses and begins to suffer diffraction losses, or even moves completely off the lenses and is lost. Therefore, it is very important to design a beam waveguide so that the light beam remains as close to the guide axis as possible.

The present paper discusses the design of intentional bends of the beam-waveguide and gives relations between the waveguide parameters such as spacing L and focal length f of the lenses and the radius of curvature of the guide axis.

The maximum amount by which the light beam deviates from the guide axis depends on the radius of curvature of the beam-waveguide. For any given radius of curvature one can influence the maximum deviation of the light ray from the guide axis by changing the spacing and focal length of the lenses. However, spacing and focal length also influences the width of the beam on the lenses. The light beam is not a mathematically thin line but has a width which is described by the spot size of the beam on the lenses.

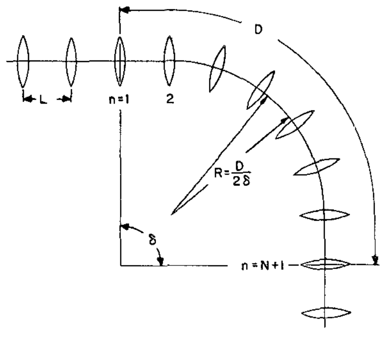

Fig. 1. Beam-waveguide bend.

Generally, sharp bends can be tolerated if the spot size is small [3]. A small spot size on the other hand requires closely spaced lenses and makes the beam-waveguide more expensive. It appears that both the radius of curvature of the sharpest bend as well as the spot size of the lowest order waveguide mode will be factors which determine the design of spacing and focal length of the lenses. If one requires that the beam shall deviate from the axis by no more than half its width then the spacing and focal length of the lenses are uniquely determined. We present curves of focal length and lens spacing as functions of the radius of curvature of the beam-waveguide for given spot sizes.

It is also possible to express the radius of curvature, which results in a beam deviation equal to half its spot size, as a function of the lens spacing and focal length only, regardless of spot size. Curves of radius of curvature and spot size as functions of the ratio of lens spacing to focal length are presented.

Random transverse displacements of the lenses also cause the light beam to depart from the guide axis [5], [6]. Again allowing the departure to equal half the spot size enables us to determine the acceptable rms displacement of the lenses. The latter goes to zero as the lens spacing approaches four times the focal length of the lenses. An actual design of a beam-waveguide will have to compromise between the requirement of leading a beam around a bend and keeping the position tolerances within reasonable limits.

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L AND f AS FUNCTIONS OF R AND w

The spot size w of the lowest order mode of the beam-waveguide is given by [4]

$$w^4 = \frac{\left(\frac{2\lambda f}{\pi}\right)^2}{\frac{4f}{L} - 1}. \tag{1}$$

- W = radius of the spot on the lens at which the intensity of the beam has dropped to e^{-2} of its maximum value.
- f = focal length of the lenses.
- L = lens spacing.
- λ = free space wavelength of the light.

If the light ray is displaced from the axis of the straight beam-waveguide it follows an oscillating trajectory given by [2]

$$r_n = A \cos (n\theta + \phi);$$

r_n is the position of the ray at the nth lens measured from its center, A and ϕ are amplitude and phase of the oscillation and θ [defined in (3b)] determines the period of oscillation.

If we assume that the light ray travels on-axis of the straight section of the beam-waveguide, it begins to depart from the axis as it enters a waveguide bend.

The deviation r_n of the light beam on a tapered bend (Fig. 1) measured at the position of the nth lens is given by (23a) of [2]:

$$r_n = \frac{2\delta L^3}{D^2 \sin^2\theta}\left\{(n-2)(1+\cos\theta) + \frac{2}{\theta}\left[\sin\theta - \sin\theta(n-1)\right]\right\}. \tag{2}$$

This equation holds for $2 \leqq n \leqq \frac{1}{2}N+2$.

The length of the bend is given by

$$D = NL, \tag{3a}$$

with N being the number of lenses on the bend and θ is defined by

$$\cos\theta = 1 - \frac{L}{2f} \tag{3b}$$

or

$$\sin\theta = \sqrt{\frac{L}{f}}\sqrt{1-\frac{L}{4f}}, \tag{3c}$$

while δ is the angle through which the bend leads.

The minimum radius of curvature R of the tapered bend is

$$R = \frac{D}{2\delta}. \tag{4}$$

The curly bracket of (2) contains two terms. One term grows proportionally with n and reaches its maximum at $n=\frac{1}{2}N+2$ at the point of lowest radius of curvature. (The lens number n was chosen so that $n=\frac{1}{2}N+2$ designates the center of the bend.) The other term oscillates with n. r_n reaches its largest possible value $r_{\max}$ if $n=\frac{1}{2}N+2$ and $\sin\theta(n-1)=-1$;

$$r_{\max} = \frac{2\delta L^3}{D^2\sin^2\theta}\left\{\frac{D}{2L}(1+\cos\theta) + \frac{2}{\theta}\left[1+\sin\theta\right]\right\}. \tag{5}$$

Using (3b) and (3c) we find that

$$\frac{1+\cos\theta}{\sin^2\theta} = \frac{2f}{L}. \tag{6}$$

If $D/L \gg 1$ and if $\sin\theta$ is not too close to zero the first term in the bracket of (5) dominates.

In that case [7]

$$r_{\max} = \frac{2\delta L f}{D} = \frac{Lf}{R}. \tag{7}$$

This is the same value by which the beam departs from the axis of a circular bend if it is launched to traverse it without oscillations [2].

If we require $r_{\max}=w$, then (7) and (1) allow us to express L and f in terms of w and R.

$$\frac{f}{\lambda} = \frac{1}{2\sqrt{\frac{\lambda}{w}\frac{\lambda}{R} - \frac{1}{\pi^2}\left(\frac{\lambda}{w}\right)^4}} \tag{8}$$

and

$$\frac{L}{\lambda} = 2\frac{R}{\lambda}\frac{w}{\lambda}\sqrt{\frac{\lambda}{w}\frac{\lambda}{R} - \frac{1}{\pi^2}\left(\frac{\lambda}{w}\right)^4}. \tag{9}$$

From (9) and (8) follows

$$\frac{L}{f} = 4 - \frac{R}{w}\left(\frac{2\lambda}{\pi w}\right)^2. \tag{10}$$

As R decreases L/f approaches 4 which, according to (3c), means that $\sin\theta$ approaches 0. The approximation leading to (8) and (10) does not hold in that limit. We have to consider another approximation for the region $L/f \approx 4$.

As L/f approaches 4 the angle θ approaches π. Since we want to study the relationship between L, f, R, and w near that region we take

$$L = 4f(1-\epsilon) \tag{11}$$

and write (5) by taking $(1+\sin\theta)/\theta \approx 1/\pi$;

$$r_{\max} \approx \frac{Lf}{R} + \frac{4\delta L^2 f}{\pi D^2 \epsilon} \approx \frac{4f^2}{R} + \frac{64\delta f^3}{\pi D^2 \epsilon}. \tag{12}$$

Equation (1) becomes

$$w^4 \approx \left(\frac{2\lambda}{\pi}\right)^2 \frac{f^2}{\epsilon}. \tag{13}$$

Eliminating ϵ from (12) and (13) yields

$$r_{\max} \approx \frac{4f^2}{R} + \frac{16\pi\delta w^4}{\lambda^2 D^2} f. \tag{14}$$

Taking $r_{\max} = w$ allows us to write a second order equation for f

$$f^2 + \frac{4\pi\delta R w^4}{\lambda^2 D^2} f - \frac{wR}{4} = 0$$

with the solution

$$f = \frac{\pi w^4}{2\delta\lambda^2 R}\left\{\sqrt{1 + \frac{R}{w}\left(\frac{\delta\lambda^2 R}{\pi w^3}\right)^2} - 1\right\} \tag{15}$$

where use was made of (4).

The equations corresponding to (8) and (9) but which are valid in the vicinity of $L = 4f$ are from (11), (13), and (15);

$$\frac{f}{\lambda} = \frac{\pi}{2\delta}\left(\frac{w}{\lambda}\right)^4 \frac{\lambda}{R} \cdot \left\{\sqrt{1 + \left(\frac{\delta}{\pi}\right)^2\left(\frac{\lambda}{w}\right)^7\left(\frac{R}{\lambda}\right)^3} - 1\right\}, \tag{16}$$

$$\frac{L}{\lambda} = 4\frac{f}{\lambda}\left\{1 - \left(\frac{2}{\pi}\frac{\lambda^2}{w^2}\frac{f}{\lambda}\right)^2\right\}. \tag{17}$$

It has to be pointed out that the approximation (7), seemingly breaks down in yet another region as $\theta \to 0$. Equation (7) was obtained from (5) by neglecting the second term in the curly bracket. This term apparently grows larger as $\theta \to 0$. However, it can be shown that (7) is still a good approximation as long as D is much larger than the period of oscillation of the light beam.

As $\theta \to 0$ we can write with the help of (3c)

$$\sin\theta \approx \theta \approx \sqrt{\frac{L}{f}}. \tag{18}$$

In order to keep the beam-waveguide effective, the limit $\sqrt{L/f} \to 0$ has to be taken by allowing Lf to remain finite. This requirement stems from the fact that the argument of $\sin\theta(n-1)$ has to remain finite in order to ensure an oscillating light beam [2].

$$\theta n = \frac{\theta}{L} \cdot nL = \frac{nL}{\sqrt{Lf}} \tag{19}$$

nL is the length coordinate measured along the waveguide axis and $\Delta = 2\pi\sqrt{Lf}$ is the oscillation period of the beam. The light waveguide remains effective as long as it forces the beam to follow an oscillating trajectory. This justifies the requirement that Δ shall remain finite. $\theta \to 0$ can, therefore, only be achieved by letting $L \to 0$ and $f \to \infty$.

Equation (5) can now be written

$$r_{\max} = \frac{2\delta Lf}{D^2}\{D + 2(L + \sqrt{fL}\} \tag{20}$$

Since $L \to 0$ and as long as $D \gg \Delta = 2\pi\sqrt{Lf}$ we can write

$$r_{\max} \approx \frac{2\delta Lf}{D} = \frac{Lf}{R}. \tag{21}$$

This proves that no new approximation is required to cover the region $\theta \to 0$ but that (8), (9), and (10) are still valid.

Discussion

Figure 2 shows L/λ and f/λ as functions of R/λ for $w/\lambda = 500$. This figure allows us to find the values of L and f which ensure that the light departs from the axis by an amount equal to w for any given radius of curvature R. Once the values of L and f are found, R can be allowed to become larger than the value originally chosen in the diagram because for larger values of R the beam will depart even less from the axis while for smaller R it will depart more. This explains why the diagram gives no values for L and f for radii R larger than certain limits (the point when $L = 0$ and $f = \infty$) because the beam will not depart as much as w from the axis no matter how we choose L and f as long as the requirement $D \gg \Delta$ is not violated.

It is important to keep in mind that w is constant for this graph. All corresponding values of L and f (those belonging to the same value R of the abscissa) belong to the beam-waveguide with the same spot size. The fact that w^4 is a quadratic function of f [1] explains why we find two values of f for each value of L or in other words why the curve for L shows a maximum. This maximum falls at the point $L/f = 2$, that is for confocal spacing of the lenses.

The figure shows an interesting fact. It is not the confocal beam-waveguide which allows us to negotiate the sharpest bends but rather one for which L/f approaches the value four even though the waveguide is unstable for $L/f = 4$ and is unable to guide a beam for $L/f > 4$. Yet, it approaches $L/f = 4$ to guide beams around the sharpest bends.

Figures 3, 4, and 5 show L/λ, f/λ, and L/f as functions of R/λ for different values of w/λ.

The point $L = 0$, $f = \infty$ moves toward smaller values of R/λ as w/λ decreases. This means that the critical value of R/λ, where the beam can depart from the axis by as much as half the spot size, decreases as w/λ decreases. It is also apparent that the confocal waveguide (maximum of L/f in Fig. 3) tolerates sharper bends if w/λ is decreased.

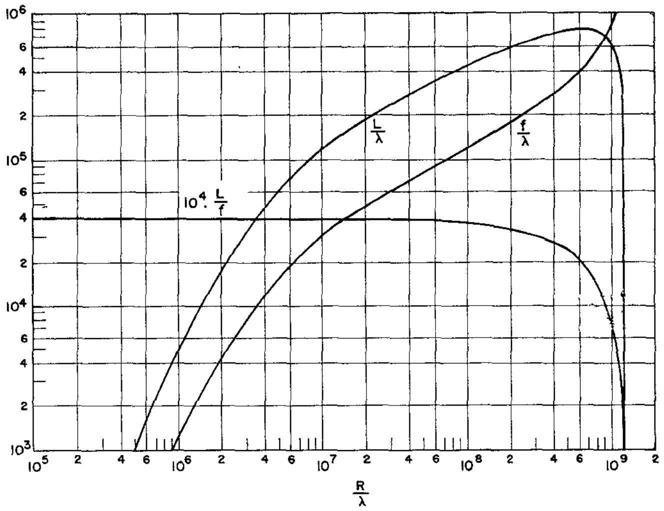

Fig. 2. Lens spacing L, focal length f and L/f as functions of the radius of curvature R of the beam-waveguide for $\delta=\pi/2$ and $w/\lambda=500$. (w is the spot size, λ the light wavelength and δ the angle of the bend.)

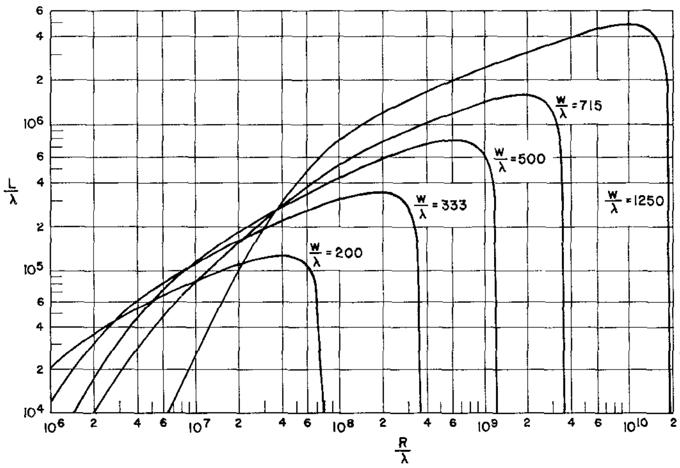

Fig. 3. Lens spacing L divided by light wavelength λ as a function of the normalized radius of curvature R for various values of the spot size w. ($\delta=\pi/2$.)

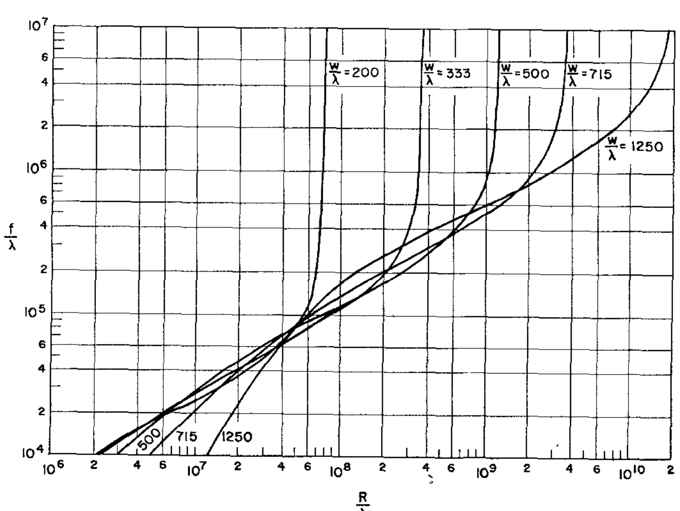

Fig. 4. Focal length f as function of R for various values of w/λ. ($\delta=\pi/2$.)

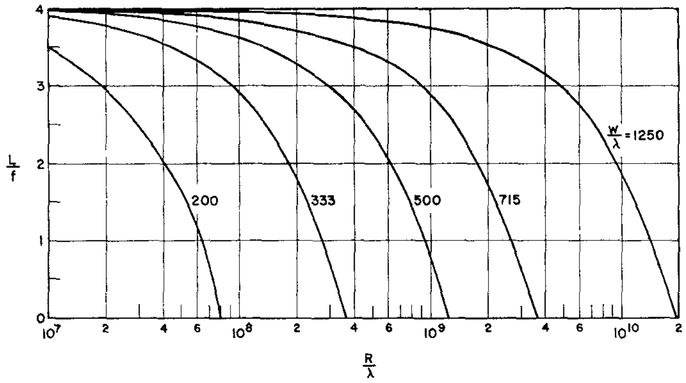

Fig. 5. Ratio of lens spacing to focal length as a function of R/λ for various values of w/λ. ($\delta=\pi/2$.)

However, if we pick a certain value of R/L in Fig. 3, say $R/\lambda=10^8$, the graph shows that the values of L/λ increase with increasing spot size. The reason for this behavior is explained if we recall that as the spot size increases the beam is also allowed to depart further from the axis. If the bending radius is too small, (e.g., $R/\lambda=10^6$), large values of L/λ can be achieved only by reducing the spot size.

Radius of Curvature and Spot Size as Functions of L/f

Since each two of the quantities R, w, L, and f can be expressed as functions of the two others it is possible to express R and w as functions of L and f.

Again requiring $r_{max}=w$ we get from (1) and (7)

$$\left(\frac{\lambda}{L}\right)^{3/2}\frac{R}{\lambda}=\left(\frac{\pi}{2}\right)^{1/2}\left(\frac{f}{L}\right)^{1/2}\left\{4\frac{f}{L}-1\right\}^{1/4} \tag{22}$$

$$\left(\frac{\lambda}{L}\right)^{1/2}\frac{w}{\lambda}=\left(\frac{2}{\pi}\right)^{1/2}\frac{\left(\frac{f}{L}\right)^{1/2}}{\left\{4\frac{f}{L}-1\right\}^{1/4}}. \tag{23}$$

Equation (22) does not hold in the limit $L\to 4f$; we have to use (12) and (4) to derive:

$$\left(\frac{\lambda}{L}\right)^{3/2}\frac{R}{\lambda}=2\left(\frac{L}{\lambda}\right)^{1/2}\left(\frac{f}{L}\right)^{2}\frac{\lambda}{w}\cdot\left\{1+\sqrt{1+\frac{4\frac{w}{\lambda}\frac{L}{f}}{\pi\delta\frac{L}{\lambda}\left(1-\frac{1}{4}\frac{L}{f}\right)}}\right\} \tag{24}$$

with w/λ of (23). Figure 6 shows $(\lambda/L)^{3/2}R/\lambda$ and $(\lambda/L)^{1/2}w/\lambda$ as functions of L/f. This figure shows clearly that R decreases for increasing values of L/f. Only in the immediate vicinity of $L/f\to 4$ does R rise again to reach $R=\infty$ at $L/f=4$. This part of the plot is the contribution of (24). The exact functional dependence of $(\lambda/L)^{3/2}R/\lambda$ in the immediate vicinity of

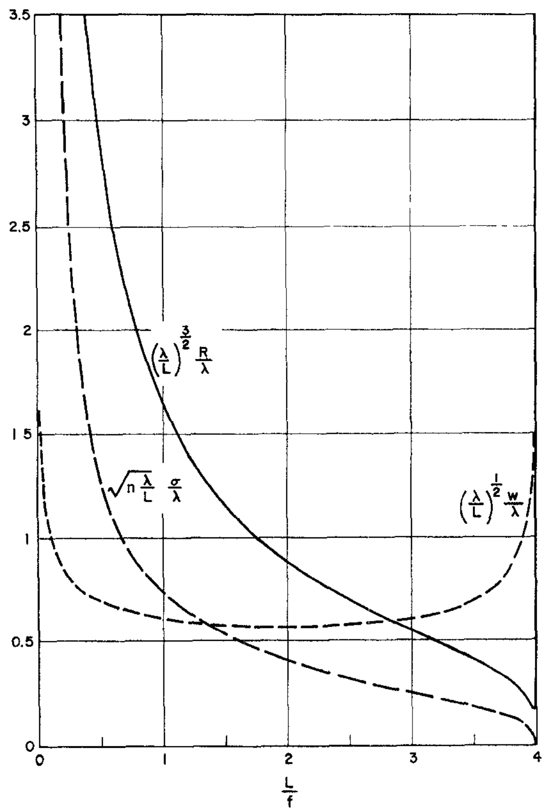

Fig. 6. Normalized radius of curvature R, spot size w and rms lens displacement σ as functions of L/f. (n = number of lenses.)

$L/f \to 4$ depends on the angle δ and on L/λ. However, the rise of the function is so steep that the details of the function in the range $3.95 < L/f < 4$ cannot be displayed in Fig. 6.

Random Lens Displacements

The discussion of the bent beam-waveguide so far came to the result that in order to negotiate sharp bends L/f should be chosen close to four but not exactly equal to four. We want to conclude this discussion with a word of caution. So far we have disregarded random displacements of the lenses. However, random transverse displacements of the lenses cause an rms deviation of the light beam from its ideal trajectory [5], [6].

Hirano and Fukatsu [6] have shown that the rms deviation of the light beam for a large number n of lenses is given by

$$\sqrt{\langle (r_n - \langle r_n \rangle)^2 \rangle} \approx \sigma\sqrt{n}\,\frac{\sqrt{2}}{\sqrt{4\frac{f}{L} - 1}} \tag{25}$$

with σ being the rms value of the transverse lens displacement and $\langle r_n \rangle$ the ensemble average.

If we again require

$$\sqrt{\langle (r_n - \langle r_n \rangle^2)^2 \rangle} = w$$

we obtain from (23) and (25)

$$\sqrt{n\frac{\lambda}{L}}\,\frac{\sigma}{\lambda} = \frac{\left(4\frac{f}{L} - 1\right)^{1/4}}{\sqrt{\pi\frac{L}{f}}}.$$

This function is also plotted in Fig. 6. We see that as R decreases with increasing values of L/f so does σ. The tolerance requirements become increasingly stringent as L/f approaches four. In practice the tolerances will set a limit to how close L/f should approach the value four.

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[7] Equation (29) of [2] contains an error. It should read

$$r_{\max} \approx \frac{\delta L^3 N}{D^2 \sin^2\theta}(1 + \cos\theta) = \frac{2\delta L}{DC}.$$